

Differential games of partial information forward-backward doubly stochastic differential equations and applications *

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Abstract This paper is concerned with a new type of differential game problems of forward-backward stochastic systems. There are three distinguishing features: Firstly, our game systems are forward-backward doubly stochastic differential equations, which is a class of more general game systems than other forward-backward stochastic game systems without doubly stochastic terms; Secondly, forward equations are directly related to backward equations at initial time, not terminal time; Thirdly, the admissible control is required to be adapted to a sub-information of the full information generated by the underlying Brownian motions. We give a necessary and a sufficient conditions for both an equilibrium point of nonzero-sum games and a saddle point of zero-sum games. Finally, we work out an example of linear-quadratic nonzero-sum differential games to illustrate the theoretical applications. Applying some stochastic filtering techniques, we obtain the explicit expression of the equilibrium point.

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Key words Stochastic differential game, Partial information, Forward-backward doubly stochastic differential equation, Equilibrium point, Stochastic filtering.

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1 Introduction

Game theory is a useful tool which helps us understand economic, social, political, and biological phenomena. Stochastic differential game problems also attract more and more research attentions, and are used widely in other social and behavioral sciences. When we study stochastic differential games of backward doubly stochastic differential equations (BDSDEs, for short), doubly stochastic Hamiltonian systems with boundary conditions appear naturally whose dynamics are described by initial coupled forward-backward doubly stochastic differential equations (FBDSDEs, for short). To illustrate this, we introduce an example of linear quadratic (LQ, for short) nonzero-sum differential games of BDSDEs with partial information which motivates us to initiate a study of stochastic differential games of initial coupled FBDSDEs with partial information. We now explain this in more detail.

Let T be a fixed constant and (Ω, \mathcal{F}, P) be a complete filtered probability space, on which two mutually independent standard Brownian motions $B(\cdot) \in \mathbb{R}^l$ and $W(\cdot) \in \mathbb{R}^d$ are defined. Let \mathcal{N} denote the class of P -null sets of \mathcal{F} . For each $t \in [0, T]$, we define

$$\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where $\mathcal{F}_t^W = \mathcal{N} \vee \sigma\{W(r) - W(0) : 0 \leq r \leq t\}$ and $\mathcal{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B(r) - B(t) : t \leq r \leq T\}$. Note that the set $\mathcal{F}_t, t \in [0, T]$ is neither increasing nor decreasing, so it does not constitute a filtration. We denote by $\mathcal{L}_T^p(\Omega; \mathcal{S})$ all class of \mathcal{F}_T -measurable random variables $\{\xi : \Omega \longrightarrow \mathcal{S}\}$ satisfying $E|\xi|^p < \infty$, by $\mathcal{L}_{\mathcal{F}_t}^p(0, T; \mathcal{S})$ all class of \mathcal{F}_t -adapted stochastic processes $\{x(t) : [0, T] \times \Omega \longrightarrow \mathcal{S}\}$ satisfying $\mathbb{E}[\int_0^T |x(t)|^p dt] < +\infty$. If there is no risk of confusion, we write $\mathcal{L}_T^p = \mathcal{L}_T^p(\Omega; \mathcal{S})$, $\mathcal{L}_{\mathcal{F}_t}^p = \mathcal{L}_{\mathcal{F}_t}^p(0, T; \mathcal{S})$. The processes $v_1(t) = v_1(t, \omega)$ and $v_2(t) = v_2(t, \omega)$ are our open-loop control processes. Let U_i be a nonempty convex subset of \mathbb{R}^{k_i} ($i = 1, 2$). In many cases in which the full information \mathcal{F}_t is inaccessible for players, ones can only observe a partial information. For this, we denote the set of all open-loop admissible controls for the player i by

$$\mathcal{U}_i = \left\{ v_i(\cdot) : [0, T] \times \Omega \longrightarrow U_i \mid v_i(\cdot) \text{ is } \mathcal{E}_t\text{-adapted and satisfies } \mathbb{E} \int_0^T |v_i(t)|^2 dt < \infty \right\},$$

where $i = 1, 2$, \mathcal{E}_t is an available sub-information of full information \mathcal{F}_t for players, i.e.

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \quad \text{for all } t.$$

For example, \mathcal{E}_t could be the δ -delayed information defined by

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+},$$

where δ is a given positive constant delay. Each element of \mathcal{U}_i is called an admissible control for Player i on $[0, T]$ ($i = 1, 2$). $\mathcal{U}_1 \times \mathcal{U}_2$ is called the set of open-loop admissible controls for the players.

We consider the following 1-dimensional linear BDSDE

$$\begin{cases} -dY(t) = [A_1 Y(t) + B_1 Z(t) + C_1 v_1(t) + D_1 v_2(t)]dt \\ \quad + [A_2 Y(t) + B_2 Z(t) + C_2 v_1(t) + D_2 v_2(t)]\hat{d}B(t) - Z(t)dW(t), \\ Y(T) = \xi, \end{cases} \quad (1)$$

and the performance criterion, for $i = 1, 2$,

$$\begin{aligned} J_i(v_1(\cdot), v_2(\cdot)) = & -\frac{1}{2}\mathbb{E}\left\{\langle F_{i1}Y(0), Y(0)\rangle + \int_0^T \left[\langle F_{i2}Y(t), Y(t)\rangle \right. \right. \\ & \left. \left. + \langle F_{i3}Z(t), Z(t)\rangle + \langle F_{i4}v_1(t), v_1(t)\rangle + \langle F_{i5}v_2(t), v_2(t)\rangle\right]dt\right\}, \end{aligned} \quad (2)$$

where the integral with respect to $\hat{d}B(t)$ is a "backward Itô integral" and the integral with respect to $dW(t)$ is a standard forward Itô integral. These are two types of particular cases of the Itô-Skorohod integral (see Nualart and Pardoux[13]). The extra noise $\{B(t)\}$ can be considered some extra information that can not be detected in practice, such as in a derivative security market, but is valuable to the partial investors.

If we let $A_2, B_2, C_2, D_2 \equiv 0$, then equation (1) is reduced to a general backward stochastic differential equation (BSDE, for short) of Pardoux-Peng's type (see Pardoux and Peng[16]). For simplicity, we assume temporarily that $\xi \in \mathcal{L}_T^2(\Omega, \mathbb{R}^1)$, all coefficients in (1) and (2) are 1-dimensional, $l = d = 1$, $F_{i1}, F_{i2}, F_{i3} \geq 0$, $F_{i4}, F_{i5} > 0$.

Our aim is to seek an equilibrium point $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, such that

$$\begin{cases} J(u_1(\cdot), u_2(\cdot)) \geq J(v_1(\cdot), u_2(\cdot)), \\ J(u_1(\cdot), u_2(\cdot)) \geq J(u_1(\cdot), v_2(\cdot)). \end{cases}$$

We call it an LQ nonzero-sum differential game of BDSDE and denote it by *Problem (LQNZB)*.

Applying Theorem 4.1 in Han et al.[4], we conclude that the equilibrium point must satisfy the following form:

$$\begin{cases} u_1(t) = -\mathbb{E}[F_{14}^{-1}(C_1 y_1(t) + C_2 z_1(t)) | \mathcal{E}_t], \\ u_2(t) = \mathbb{E}[F_{25}^{-1}(D_1 y_2(t) + D_2 z_2(t)) | \mathcal{E}_t], \end{cases} \quad (3)$$

where (y_i, z_i) ($i = 1, 2$) is the solution of the following initial coupled FBDSDE:

$$\left\{ \begin{aligned} -dY(t) &= \left[A_1 Y(t) + B_1 Z(t) - C_1 F_{14}^{-1}(C_1 y_1(t) + C_2 z_1(t)) \right. \\ &\quad \left. - D_1 F_{25}^{-1}(D_1 y_2(t) + D_2 z_2(t)) \right] dt \\ &\quad + \left[A_2 Y(t) + B_2 Z(t) - C_2 F_{14}^{-1}(C_1 y_1(t) + C_2 z_1(t)) \right. \\ &\quad \left. - D_2 F_{25}^{-1}(D_1 y_2(t) + D_2 z_2(t)) \right] \hat{d}B(t) - Z(t) dW(t), \\ dy_i(t) &= (A_1 y_i(t) + A_2 z_i(t) + F_{i2} Y(t)) dt + (B_1 y_i(t) + B_2 z_i(t) + F_{i3} Z(t)) dW(t) \\ &\quad - z_i(t) \hat{d}B(t), \\ Y(T) &= \xi, \quad y_i(0) = F_{i1} Y(0). \end{aligned} \right. \quad (4)$$

We see that the equation for $Y(\cdot)$ is backward (since it is given the final datum which is an \mathcal{F}_T -measurable random variable), the equation for $y_i(\cdot)$ is forward (since it is given the initial datum which is directly related with the backward solution $Y(\cdot)$ at initial time). Further, the backward equation is “forward” with respect to the backward stochastic integral $\hat{d}B(t)$, as well as “backward” with respect to the forward stochastic integral $dW(t)$; the coupled forward equation is “backward” with respect to the backward stochastic integral $\hat{d}B(t)$, as well as “forward” with respect to the forward stochastic integral $dW(t)$. Equation (4) is exactly the type of time-symmetric forward-backward stochastic differential equations (FBSDEs, for short) introduced by Peng and Shi[19]. There is a small difference between equation (4) and FBSDE in Peng and Shi[19]: the former is initial coupled, but the latter is terminal coupled. So we call equation (4) an initial coupled linear FBDSDE. In addition, the candidate equilibrium point denoted by (3) involves the available sub-information \mathcal{E}_t of full information \mathcal{F}_t for players. Thus, a type of initial coupled FBDSDE naturally appears when we study *Problem (LQNZB)*. Since this type of FBDSDEs possess fine dynamics and can be reduced to FBSDEs or BDSDEs or BSDEs, one could not help thinking about differential game problems for initial coupled FBDSDEs under partial information.

Pardoux[15] generalized the classical Feynman-Kac formula and provided a probabilistic representation for solutions of linear parabolic stochastic partial differential equations (SPDEs, for short). By introducing originally BDSDEs, which is a new class of BSDEs and covers the results of Pardoux and Peng[16], Pardoux and Peng[17] produced a probabilistic representation of certain quasi-linear SPDEs as an extension to the Feynman-Kac formula for linear SPDEs. In general, that a forward SDE of Itô's type couples a backward SDE of Pardoux-Peng's type, which maybe couple each other at initial conditions or terminal conditions, constitutes an initial or terminal coupled FBSDE. The theory of FBSDEs has received considerable research attention in recent years. For more information on the solvability of FBSDEs and corresponding optimal control problems with full or partial information, see e.g. Antonelli[1], Hu and Peng[5], Ma, Protter and Yong[9], Meng[11], Øksendal and Sulem[14], Peng and Shi[18], Peng and Wu[20], Shi and Wu[21, 22], Wang and Wu[23, 24], Wu[27, 28], specially the monographs by Ma and Yong[10] and Yong and Zhou[32], etc.

There is a few literature on differential games of BSDEs and FBSDEs. Yu and Ji[34] obtained an existence and uniqueness result for an initial coupled FBSDE under some monotone conditions, applied it to backward linear-quadratic nonzero-sum stochastic differential game problem and got the explicit form of a Nash equilibrium point. Wang and Yu[25] established a necessary and a sufficient conditions for an equilibrium point of nonzero-sum differential game of BSDEs and applied them to study a financial problem. Zhang[35] extended the result of Yu and Ji[34] to the case where BSDEs are driven by both Brownian motion and Poisson random measure. Wang and Yu[26] recently generalize the results of [25] to partial information differential games and obtain the corresponding maximum principle and verification theorem, and they also apply the theoretical results to study LQ differential games and financial problem. Yu[33] mainly studied the LQ optimal control and nonzero-sum differential game of FBSDE. Hui and Xiao[7] investigated differential games of FBSDEs, and established the maximum principle and verification theorem for both an equilibrium point of nonzero-sum cases and a saddle point of zero-sum cases. Meng[12] discussed the partial information zero-sum differential games of fully coupled FBSDEs.

Han et al.[4] investigated the optimal control for BDSDEs and obtain a stochastic maximum

principle of the optimal control. In [19], Peng and Shi established the existence and uniqueness results of terminal coupled FBDSDEs under certain monotonicity assumptions. Zhu et al.[37] relaxed the monotonicity assumptions and allowed the case of different dimensions between forward equations and backward equations, compared with the results in Peng and Shi[19]. Zhang and Shi[36] studied the optimal control of fully terminal-coupled FBDSDEs and obtained the maximum principle in the global form, where the control variables can enter into the diffusion coefficients and the control domain need not be convex. Using the solution of FBDSDEs, Zhang and Shi also got the explicit form of open-loop Nash equilibrium point for nonzero-sum stochastic differential games of only forward doubly stochastic differential equations.

However, none of the works mentioned above deals with differential games of initial coupled FBDSDEs with full information or partial information. In Section 2, we formulate the zero-sum and nonzero-sum games of initial-coupled FBDSDEs with partial information. In Section 3, we are devoted to proving a maximum principle and a verification theorem for both an equilibrium point of nonzero-sum games and a saddle point of zero-sum games. In Section 4, an example of a nonzero-sum differential game is worked out to illustrate theoretical applications. In terms of maximum principle and verification theorem, the explicit expression of an equilibrium point is obtained. Finally, we give some concluding remarks.

2 Formulation of the problem

We introduce the mappings

$$\begin{aligned}
f &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times U_1 \times U_2 \rightarrow \mathbb{R}^n, \\
\bar{f} &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times U_1 \times U_2 \rightarrow \mathbb{R}^{n \times d}, \\
g &: [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times U_1 \times U_2 \rightarrow \mathbb{R}^m, \\
\bar{g} &: [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times U_1 \times U_2 \rightarrow \mathbb{R}^{m \times l}, \\
\phi &: \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad \varphi, \varphi_i : \mathbb{R}^m \rightarrow \mathbb{R}^1, \quad \gamma, \gamma_i : \mathbb{R}^n \rightarrow \mathbb{R}^1, \\
l, l_i &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times U_1 \times U_2 \rightarrow \mathbb{R}^1 \ (i = 1, 2).
\end{aligned}$$

Assumption (H1): For any $(y, z, Y, Z, v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times U_1 \times U_2$, we assume

that

$$\begin{aligned} f(\cdot, y, z, Y, Z, v_1, v_2) &\in \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^n), & \bar{f}(\cdot, y, z, Y, Z, v_1, v_2) &\in \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^{n \times d}), \\ g(\cdot, Y, Z, v_1, v_2) &\in \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^m), & \bar{g}(\cdot, Y, Z, v_1, v_2) &\in \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^{m \times l}). \end{aligned}$$

We assume moreover that f, \bar{f}, g and \bar{g} are continuously differentiable with respect to (y, z, Y, Z, v_1, v_2) and their derivatives with respect to (y, z, Y, Z, v_1, v_2) are continuous and uniformly bounded. $l, l_1, l_2, \varphi, \varphi_1, \varphi_2, \gamma, \gamma_1$ and γ_2 are continuously differential with respect to (y, z, Y, Z, v_1, v_2) and their derivatives with respect to (y, z, Y, Z, v_1, v_2) are continuous and bounded by $K(1 + |y| + |z| + |Y| + |Z| + |v_1| + |v_2|)$. There exists constants $k > 0$ and $0 < c < 1$ such that

$$\begin{aligned} &|\bar{f}(t, y_1, z_1, Y_1, Z_1, u_1, u_2) - \bar{f}(t, y_2, z_2, Y_2, Z_2, v_1, v_2)|^2 \\ &\leq k(|y_1 - y_2|^2 + |Y_1 - Y_2|^2 + |Z_1 - Z_2|^2 + |u_1 - v_1|^2 + |u_2 - v_2|^2) + c|z_1 - z_2|^2, \\ &|\bar{g}(t, Y_1, Z_1, u_1, u_2) - \bar{g}(t, Y_2, Z_2, v_1, v_2)|^2 \\ &\leq k(|Y_1 - Y_2|^2 + |u_1 - v_1|^2 + |u_2 - v_2|^2) + c|Z_1 - Z_2|^2, \end{aligned}$$

for all $(y_1, z_1, Y_1, Z_1, u_1, u_2), (y_2, z_2, Y_2, Z_2, v_1, v_2) \in \mathcal{R}^n \times \mathcal{R}^{n \times l} \times \mathcal{R}^m \times \mathcal{R}^{m \times d} \times U_1 \times U_2$.

In the following, we specify the problems of nonzero-sum and zero-sum differential games of forward-backward doubly stochastic systems, respectively. For simplicity, we denote them by *Problem (NZSG)* and *Problem (ZSG)*, respectively.

Consider an FBDSDE

$$\left\{ \begin{aligned} -dY^{v_1, v_2}(t) &= g(t, Y^{v_1, v_2}(t), Z^{v_1, v_2}(t), v_1(t), v_2(t))dt \\ &\quad + \bar{g}(t, Y^{v_1, v_2}(t), Z^{v_1, v_2}(t), v_1(t), v_2(t))\hat{d}B(t) - Z^{v_1, v_2}(t)dW(t), \\ dy^{v_1, v_2}(t) &= f(t, y^{v_1, v_2}(t), z^{v_1, v_2}(t), Y^{v_1, v_2}(t), Z^{v_1, v_2}(t), v_1(t), v_2(t))dt \\ &\quad + \bar{f}(t, y^{v_1, v_2}(t), z^{v_1, v_2}(t), Y^{v_1, v_2}(t), Z^{v_1, v_2}(t), v_1(t), v_2(t))dW(t) \\ &\quad - z^{v_1, v_2}(t)\hat{d}B(t), \\ Y^{v_1, v_2}(T) &= \xi, \quad y^{v_1, v_2}(0) = \phi(Y^{v_1, v_2}(0)), \quad 0 \leq t \leq T. \end{aligned} \right. \quad (5)$$

Under the assumption (H1), there exists a unique solution $(y^{v_1, v_2}(\cdot), z^{v_1, v_2}(\cdot), Y^{v_1, v_2}(\cdot), Z^{v_1, v_2}(\cdot)) \in \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^n) \times \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^{n \times l}) \times \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^m) \times \mathcal{L}_{\mathcal{F}_t}^2(0, T; \mathcal{R}^{m \times d})$ to equation (5) for any $(v_1(\cdot), v_2(\cdot))$

$\in \mathcal{U}_1 \times \mathcal{U}_2$ (see Pardoux and Peng[17]). In the case where equation (5) does not involve the term of backward Itô's integral, i.e. $\bar{g} \equiv 0$, f and \bar{f} are independent of z^{v_1, v_2} , the game systems will be reduced to the initial coupled FBSDEs which has been studied by Xiao and Wang[30]. In the case where equation (5) does not involve the forward equation, i.e. $f = \bar{f} = \phi \equiv 0$, the game systems will be reduced to the BDSDEs which has been investigated by Han et al.[4]. In the case where equation (5) does not involve both the term of backward Itô's integral and the forward equation, the game systems will be reduced to the BSDEs which has been investigated by Wang and Yu[25, 26] and Yu and Ji[34].

Consider a performance criterion

$$J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E} \left[\int_0^T l_i(t, y^{v_1, v_2}(t), z^{v_1, v_2}(t), Y^{v_1, v_2}(t), Z^{v_1, v_2}(t), v_1(t), v_2(t)) dt \right. \\ \left. + \varphi_i(Y^{v_1, v_2}(0)) \right] + \gamma_i(y^{v_1, v_2}(T)) \quad (6)$$

with $l_i(\cdot, y^{v_1, v_2}(\cdot), z^{v_1, v_2}(\cdot), Y^{v_1, v_2}(\cdot), Z^{v_1, v_2}(\cdot), v_1(\cdot), v_2(\cdot)) \in \mathcal{L}_{\mathcal{F}_t}^1(0, T; \mathbb{R})$ and $\varphi_i \in \mathcal{L}^1(0, T; \mathbb{R})$ for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ ($i = 1, 2$), and

$$J(v_1(\cdot), v_2(\cdot)) = \mathbb{E} \left[\int_0^T l(t, y^{v_1, v_2}(t), z^{v_1, v_2}(t), Y^{v_1, v_2}(t), Z^{v_1, v_2}(t), v_1(t), v_2(t)) dt \right. \\ \left. + \varphi(y^{v_1, v_2}(T)) \right] + \gamma(Y^{v_1, v_2}(0)) \quad (7)$$

with $l(\cdot, y^{v_1, v_2}(\cdot), z^{v_1, v_2}(\cdot), Y^{v_1, v_2}(\cdot), Z^{v_1, v_2}(\cdot), v_1(\cdot), v_2(\cdot)) \in \mathcal{L}_{\mathcal{F}_t}^1(0, T; \mathbb{R})$ and $\varphi \in \mathcal{L}^1(0, T; \mathbb{R})$ for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. We note that (6) and (7) are well posed. There are two players i_1 and i_2 . Player i_1 controls v_1 and Player i_2 controls v_2 .

Problem (NZSG): Find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) \geq J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) \geq J_2(u_1(\cdot), v_2(\cdot)), \end{cases} \quad (8)$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. We call $(u_1(\cdot), u_2(\cdot))$ an open-loop equilibrium point of *Problem (NZSG)* (if it does exist). It is easy to see that the existence of an open-loop equilibrium point implies

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \sup_{v_1(\cdot) \in \mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \sup_{v_2(\cdot) \in \mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)). \end{cases}$$

Problem (ZSG): Find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)), \quad (9)$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. We call $(u_1(\cdot), u_2(\cdot))$ an open-loop saddle point of *Problem (ZSG)* (if it exists). In fact the existence of an open-loop saddle point implies

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) &= \sup_{v_2(\cdot) \in \mathcal{U}_2} \left(\inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), v_2(\cdot)) \right) \\ &= \inf_{v_1(\cdot) \in \mathcal{U}_1} \left(\sup_{v_2(\cdot) \in \mathcal{U}_2} J(v_1(\cdot), v_2(\cdot)) \right). \end{aligned}$$

We shall verify this point in Theorem 3.5 (iii).

3 Differential games of FBDSDEs

3.1 Nonzero-sum case

Suppose $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem (NZSG)* with the trajectory $(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot))$ of (5). For all $t \in [0, T]$, let $v_i(t) \in U_i$ be such that $u_i(\cdot) + v_i(\cdot) \in \mathcal{U}_i$ ($i = 1, 2$).

Notice that \mathcal{U}_i is convex, then for $0 \leq \epsilon, \rho \leq 1, i = 1, 2$,

$$u_{1\epsilon}(t) = u_1(t) + \epsilon v_1(t) \in \mathcal{U}_1, \quad u_{2\rho}(t) = u_2(t) + \rho v_2(t) \in \mathcal{U}_2, \quad 0 \leq t \leq T.$$

For simplicity, we denote

$$\begin{aligned} f(t) &= f(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t)), \\ g(t) &= g(t, Y(t), Z(t), u_1(t), u_2(t)), \\ Y^{u_{1\epsilon}}(t) &= Y^{(u_1 + \epsilon v_1, u_2)}(t), \quad Y^{u_{2\rho}}(t) = Y^{(u_1, u_2 + \rho v_2)}(t), \\ h_i(\epsilon, \rho) &= J_i(u_1 + \epsilon v_1, u_2 + \rho v_2), \end{aligned}$$

define the processes

$$\hat{Y}^1(t) = \frac{d}{d\epsilon} Y^{u_{1\epsilon}}(t)|_{\epsilon=0}, \quad \hat{Y}^2(t) = \frac{d}{d\rho} Y^{u_{2\rho}}(t)|_{\rho=0},$$

and make the similar notations for $\bar{f}, \bar{g}, l_i, \hat{y}^i, \hat{z}^i, \hat{Z}^i, i = 1, 2$. For $i = 1, 2$, we have the following variational equations:

$$\begin{cases} -d\hat{Y}^i(t) = \hat{g}^i(t)dt + \hat{\bar{g}}^i(t)\hat{d}B(t) - \hat{Z}^i(t)dW(t), \\ d\hat{y}^i(t) = \hat{f}^i(t)dt + \hat{\bar{f}}^i(t)dW(t) - \hat{z}^i(t)\hat{d}B(t), \\ \hat{Y}^i(T) = 0, \hat{y}^i(0) = \phi_Y(Y(0))\hat{Y}^i(0) \end{cases}$$

where

$$\begin{aligned} \hat{g}^i(t) &= g_Y(t)\hat{Y}^i(t) + g_Z(t)\hat{Z}^i(t) + g_{v_i}(t)v_i(t), \\ \hat{\bar{g}}^i(t) &= \bar{g}_Y(t)\hat{Y}^i(t) + \bar{g}_Z(t)\hat{Z}^i(t) + \bar{g}_{v_i}(t)v_i(t), \\ \hat{f}^i(t) &= f_y(t)\hat{y}^i(t) + f_z(t)\hat{z}^i(t) + f_Y(t)\hat{Y}^i(t) + f_Z(t)\hat{Z}^i(t) + f_{v_i}(t)v_i(t), \\ \hat{\bar{f}}^i(t) &= \bar{f}_y(t)\hat{y}^i(t) + \bar{f}_z(t)\hat{z}^i(t) + \bar{f}_Y(t)\hat{Y}^i(t) + \bar{f}_Z(t)\hat{Z}^i(t) + \bar{f}_{v_i}(t)v_i(t), \\ \hat{l}^i(t) &= l_{iy}(t)\hat{y}^i(t) + l_{iz}(t)\hat{z}^i(t) + l_{iY}(t)\hat{Y}^i(t) + l_{iZ}(t)\hat{Z}^i(t) + l_{iv_i}(t)v_i(t). \end{aligned}$$

Next, we define the generalized *Hamiltonian function* $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{U}_1 \times \mathbb{U}_2 \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ as follows:

$$\begin{aligned} H_i(t, y, z, Y, Z, v_1, v_2, p_i, \bar{p}_i, q_i, \bar{q}_i) &\triangleq \langle q_i, f(y, z, Y, Z, v_1, v_2) \rangle + \langle \bar{q}_i, \bar{f}(y, z, Y, Z, v_1, v_2) \rangle \\ &\quad - \langle p_i, g(Y, Z, v_1, v_2) \rangle - \langle \bar{p}_i, \bar{g}(Y, Z, v_1, v_2) \rangle + l_i(y, z, Y, Z, v_1, v_2). \end{aligned} \quad (10)$$

Let $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the solution $(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot))$ of equation (5). We shall use the abbreviated notation $H_i(t)$ defined by

$$H_i(t) \equiv H_i(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_i(t), \bar{p}_i(t), q_i(t), \bar{q}_i(t)).$$

The adjoint equations are described by the following generalized stochastic Hamiltonian systems:

$$\begin{cases} dp_i(t) = -H_{iY}^*(t)dt - H_{iZ}^*(t)dW(t) - \bar{p}_i(t)\hat{d}B(t), \\ -dq_i(t) = H_{iy}^*(t)dt + H_{iz}^*(t)\hat{d}B(t) - \bar{q}_i(t)dW(t), \\ p_i(0) = -\phi_{iY}^*(Y(0)) - \phi_Y^*(Y(0))q_i(0), \\ q_i(T) = \gamma_{iy}^*(y(T)). \end{cases} \quad (11)$$

Then we have the following maximum principle for nonzero-sum differential games.

Theorem 3.1 (Maximum principle for nonzero-sum games) *Let (H1) hold and $(u_1(\cdot), u_2(\cdot))$ be an equilibrium point of Problem (NZSG) with the corresponding solutions $(x(\cdot), y(\cdot), z(\cdot))$ and $(p_i(\cdot), q_i(\cdot), k_i(\cdot))$ of (5) and (11). Then it follows that*

$$\left\langle E[H_{1v_1}^*(t)|\mathcal{E}_t], v_1(t) - u_1(t) \right\rangle \leq 0 \quad (12)$$

and

$$\left\langle E[H_{2v_2}^*(t)|\mathcal{E}_t], v_2(t) - u_2(t) \right\rangle \leq 0 \quad (13)$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, a.e. a.s.

Proof: Since $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point, we have

$$\frac{\partial h_1}{\partial \epsilon}(0, 0) = \lim_{\epsilon \rightarrow 0} \frac{J_1(u_1 + \epsilon v_1, u_2) - J_1(u_1, u_2)}{\epsilon} \leq 0.$$

Then

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial \epsilon} h_1(\epsilon, 0)|_{\epsilon=0} \\ &= E \int_0^T \left(l_{1y}(t) \hat{y}^1(t) + l_{1z}(t) \hat{z}^1(t) + l_{1Y}(t) \hat{Y}^1(t) + l_{1Z}(t) \hat{Z}^1(t) + l_{1v_1}(t) v_1(t) \right) dt \\ &\quad + E \left(\varphi_{1Y}(Y(0)) \hat{Y}^1(0) + \gamma_{1y}(y(T)) \hat{y}^1(T) \right). \end{aligned} \quad (14)$$

Applying Itô's formula to $\langle p_1(t), \hat{Y}^1(t) \rangle$ and $\langle q_1(t), \hat{y}^1(t) \rangle$, and integrating from 0 to T , we have

$$\begin{aligned} &E \left(\varphi_{1Y}(Y(0)) \hat{Y}^1(0) \right) \\ &= -E \langle p_1(0) + \phi_Y^*(Y(0)) q_1(0), \hat{Y}^1(0) \rangle = -\langle \phi_Y^*(Y(0)) q_1(0), \hat{Y}^1(0) \rangle \\ &\quad - E \int_0^T \left(p_1^*(t) g_{v_1}(t) v_1(t) + q_1^*(t) f_Y(t) \hat{Y}^1(t) + \bar{q}_1^*(t) \bar{f}_Y(t) \hat{Y}^1(t) - l_{1Y}(t) \hat{Y}^1(t) \right. \\ &\quad \left. + \bar{p}_1^*(t) \bar{g}_{v_1}(t) v_1(t) + q_1^*(t) f_Z(t) \hat{Z}^1(t) + \bar{q}_1^*(t) \bar{f}_Z(t) \hat{Z}^1(t) - l_{1Z}(t) \hat{Z}^1(t) \right) dt, \end{aligned} \quad (15)$$

and

$$\begin{aligned} &E \left(\gamma_{1y}(y(T)) \hat{y}^1(T) \right) = \langle \phi_Y^*(Y(0)) q_1(0), \hat{Y}^1(0) \rangle \\ &\quad + E \int_0^T \left(q_1^*(t) f_Y(t) \hat{Y}^1(t) + q_1^*(t) f_Z(t) \hat{Z}^1(t) + q_1^*(t) f_{v_1}(t) v_1(t) - l_{1y}(t) \hat{y}^1(t) \right. \\ &\quad \left. - l_{1z}(t) \hat{z}^1(t) + \bar{q}_1^*(t) \bar{f}_Y(t) \hat{Y}^1(t) + \bar{q}_1^*(t) \bar{f}_Z(t) \hat{Z}^1(t) + \bar{q}_1^*(t) \bar{f}_{v_1}(t) v_1(t) \right) dt. \end{aligned} \quad (16)$$

Substituting (15) and (16) into (14), for all $v_1 \in U_1$ such that $u_1(\cdot) + v_1(\cdot) \in \mathcal{U}_1$, we get

$$\begin{aligned}
0 &\geq \frac{\partial}{\partial \epsilon} h_1(\epsilon, 0)|_{\epsilon=0} \\
&= E \int_0^T \left(q_1^*(t) f_{v_1}(t) + \bar{q}_1^*(t) \bar{f}_{v_1}(t) + p_1^*(t) g_{v_1}(t) + \bar{p}_1^*(t) \bar{g}_{v_1}(t) + l_{1v_1}(t) \right) v_1(t) dt \\
&= E \int_0^T \left\langle H_{1v_1}^*(t), v_1(t) \right\rangle dt = E \int_0^T E \left[\left\langle H_{1v_1}^*(t), v_1(t) \right\rangle \middle| \mathcal{E}_t \right] dt,
\end{aligned} \tag{17}$$

which implies that (12) is true. The result (13) can be proved by the same method as shown in proving (12). \square

If the control process $(v_1(\cdot), v_2(\cdot))$ is admissible adapted to the filtration \mathcal{F}_t , we have the following corollary.

Corollary 3.1 (Maximum principle for full information nonzero-sum games) *Suppose that $\mathcal{E}_t = \mathcal{F}_t$ for all t . Let (H1) hold and $(u_1(\cdot), u_2(\cdot))$ be an equilibrium point of nonzero-sum differential games with the corresponding solutions $(x(\cdot), y(\cdot), z(\cdot))$ and $(p_i(\cdot), q_i(\cdot), k_i(\cdot))$ of (5) and (11). Then it follows that*

$$\left\langle H_{1v_1}^*(t), v_1(t) - u_1(t) \right\rangle \leq 0$$

and

$$\left\langle H_{2v_2}^*(t), v_2(t) - u_2(t) \right\rangle \leq 0$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, a.e. a.s.

In what follows, we proceed to establish a verification theorem, also called a sufficient condition, for an equilibrium point. For this, we introduce an additional condition as follows:

(H2) $\phi(Y) = MY$ where M is a non-zero constant matrix with order $n \times m$. φ_i and γ_i are concave in Y and y ($i = 1, 2$), respectively.

Theorem 3.2 (Verification theorem for nonzero-sum games) *Let (H1) and (H2) hold. Let $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ be with the corresponding solutions (y, z, Y, Z) and $(p_i, \bar{p}_i, q_i, \bar{q}_i)$ of*

equations (5) and (11). Suppose

$$\begin{aligned}\hat{H}_1(t, a, b, c, d) &= \sup_{v_1 \in U_1} H_1(t, a, b, c, d, v_1, u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)), \\ \hat{H}_2(t, a, b, c, d) &= \sup_{v_2 \in U_2} H_2(t, a, b, c, d, u_1(t), v_2, p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t))\end{aligned}\quad (18)$$

exist for all $(t, a, b, c, d) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, and are concave in (a, b, c, d) for all $t \in [0, T]$ (the Arrow condition).

Moreover

$$\begin{aligned}\mathbb{E} \left[H_1(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)) \middle| \mathcal{E}_t \right] \\ = \sup_{v_1 \in U_1} \mathbb{E} \left[H_1(t, y(t), z(t), Y(t), Z(t), v_1, u_2(t), p_1(t), q_1(t), \bar{q}_1(t)) \middle| \mathcal{E}_t \right],\end{aligned}\quad (19)$$

$$\begin{aligned}\mathbb{E} \left[H_2(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)) \middle| \mathcal{E}_t \right] \\ = \sup_{v_2 \in U_2} \mathbb{E} \left[H_2(t, y(t), z(t), Y(t), Z(t), u_1(t), v_2, p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)) \middle| \mathcal{E}_t \right].\end{aligned}\quad (20)$$

Then $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of Problem (NZSG).

Proof : Let $(v_1(\cdot), u_2(\cdot))$ and $(u_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the corresponding solutions $(y^{v_1}, z^{v_1}, Y^{v_1}, Z^{v_1})$ and $(y^{v_2}, z^{v_2}, Y^{v_2}, Z^{v_2})$ to equation (5). We define the following terms

$$\begin{aligned}H_1(t) &= H_1(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)), \\ H_1^{v_1}(t) &= H_1(t, y^{v_1}(t), z^{v_1}(t), Y^{v_1}(t), Z^{v_1}(t), v_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)), \\ H_1^{v_2}(t) &= H_1(t, y^{v_2}(t), z^{v_2}(t), Y^{v_2}(t), Z^{v_2}(t), u_1(t), v_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)), \\ f^{v_1}(t) &= f(t, y^{v_1}(t), z^{v_1}(t), Y^{v_1}(t), Z^{v_1}(t), v_1(t), u_2(t)), \\ f^{v_2}(t) &= f(t, y^{v_2}(t), z^{v_2}(t), Y^{v_2}(t), Z^{v_2}(t), u_1(t), v_2(t)),\end{aligned}$$

and similar notations are made for $\bar{f}^{v_1}, \bar{f}^{v_2}, \dots$.

By virtue of the concavity property of φ_1 and γ_1 , we have for $\forall v_1(\cdot) \in \mathcal{U}_1$

$$J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \leq I_1 + I_2 + I_3 \quad (21)$$

with

$$\begin{aligned} I_1 &= \mathbb{E} [\gamma_{1y}(y(T))(y^{v_1}(T) - y(T))], \\ I_2 &= \mathbb{E} [\varphi_{1Y}(Y(0))(Y^{v_1}(0) - Y(0))], \\ I_3 &= \mathbb{E} \int_0^T \left(l_1^{v_1}(t) - l_1(t) \right) dt. \end{aligned}$$

Applying Itô's formula to $\langle q_1(t), y^{v_1}(t) - y(t) \rangle$ and $\langle p_1(t), Y^{v_1}(t) - Y(t) \rangle$,

$$\begin{aligned} I_1 &= \mathbb{E}[\langle q_1(0), M(Y^{v_1}(0) - Y(0)) \rangle] \\ &\quad + \mathbb{E} \int_0^T \left(\langle q_1(t), f^{v_1}(t) - f(t) \rangle - \langle H_{1y}^*(t), y^{v_1}(t) - y(t) \rangle \right. \\ &\quad \left. + \langle \bar{q}_1(t), \bar{f}^{v_1}(t) - \bar{f}(t) \rangle - \langle H_{1z}^*(t), z^{v_1}(t) - z(t) \rangle \right) dt, \end{aligned} \quad (22)$$

$$\begin{aligned} I_2 &= -\mathbb{E}[\langle q_1(0), M(Y^{v_1}(0) - Y(0)) \rangle] \\ &\quad - \mathbb{E} \int_0^T \left(\langle p_1(t), g^{v_1}(t) - g(t) \rangle + \langle H_{1Y}^*(t), Y^{v_1}(t) - Y(t) \rangle \right. \\ &\quad \left. + \langle \bar{p}_1(t), \bar{g}^{v_1}(t) - \bar{g}(t) \rangle + \langle H_{1Z}^*(t), Z^{v_1}(t) - Z(t) \rangle \right) dt, \end{aligned} \quad (23)$$

$$\begin{aligned} I_3 &= \mathbb{E} \int_0^T \left(H_1^{v_1}(t) - H_1(t) - \langle q_1(t), f^{v_1}(t) - f(t) \rangle - \langle \bar{q}_1(t), \bar{f}^{v_1}(t) - \bar{f}(t) \rangle \right. \\ &\quad \left. + \langle \bar{p}_1(t), \bar{g}^{v_1}(t) - \bar{g}(t) \rangle + \langle p_1(t), g^{v_1}(t) - g(t) \rangle \right) dt. \end{aligned} \quad (24)$$

Substituting (22)–(24) into (21), it follows immediately that

$$\begin{aligned} &J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \\ &\leq \mathbb{E} \int_0^T \left(H_1^{v_1}(t) - H_1(t) - \langle H_{1Y}^*(t), Y^{v_1}(t) - Y(t) \rangle - \langle H_{1Z}^*(t), Z^{v_1}(t) - Z(t) \rangle \right. \\ &\quad \left. - \langle H_{1y}^*(t), y^{v_1}(t) - y(t) \rangle - \langle H_{1z}^*(t), z^{v_1}(t) - z(t) \rangle \right) dt. \end{aligned} \quad (25)$$

Since $v_1 \longrightarrow \mathbb{E} \left[H_1(t, y(t), z(t), Y(t), Z(t), v_1, u_2(t), p_1(t), q_1(t), \bar{q}_1(t)) \middle| \mathcal{E}_t \right]$ is maximum for $v_1 = u_1$ and since $v_1(t), u_1(t)$ are \mathcal{E}_t -measurable, we get

$$\begin{aligned} &\mathbb{E} \left[\frac{\partial}{\partial v_1} H_1(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)) (v_1(t) - u_1(t)) \middle| \mathcal{E}_t \right] \\ &= \mathbb{E} \left[\frac{\partial}{\partial v_1} H_1(t, y(t), z(t), Y(t), Z(t), v_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)) \middle| \mathcal{E}_t \right]_{v_1=u_1} (v_1(t) - u_1(t)) \\ &\leq 0. \end{aligned}$$

By the equality (19) and the concavity of \hat{H}_1 , we conclude that

$$J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \leq 0, \quad (26)$$

for all $v_1(\cdot) \in \mathcal{U}_1$. Repeating the similar proceeding as shown in deriving (26), we can prove that

$$J_2(u_1(\cdot), v_2(\cdot)) - J_2(u_1(\cdot), u_2(\cdot)) \leq 0. \quad (27)$$

Based on the arguments above, $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem (NZSG)*. \square

Corollary 3.2 (Verification theorem for full information nonzero-sum games) *Suppose that $\mathcal{E}_t = \mathcal{F}_t$ for all t and that (H1), (H2) and (18) hold. Let $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ be with the corresponding solutions (y, z, Y, Z) and $(p_i, \bar{p}_i, q_i, \bar{q}_i)$ of equations (5) and (11). Moreover*

$$\begin{aligned} & H_1(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)) \\ &= \sup_{v_1 \in U_1} H_1(t, y(t), z(t), Y(t), Z(t), v_1, u_2(t), p_1(t), q_1(t), \bar{q}_1(t)) \\ & H_2(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)) \\ &= \sup_{v_2 \in U_2} H_2(t, y(t), z(t), Y(t), Z(t), u_1(t), v_2, p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)). \end{aligned}$$

Then $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of nonzero-sum differential games.

3.2 Zero-sum case

In this section, we consider zero-sum differential games of FBDSDEs. In fact, zero-sum games can be consider a special case of nonzero-sum games. By the maximum principle of nonzero-sum games in Section 3.1, we can deduce the necessary conditions for a saddle point of zero-sum games. We shall detail this as follows.

Let

$$-J_1 = J_2 = J.$$

If $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem (NZSG)*, we have

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) \geq J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) \geq J_2(u_1(\cdot), v_2(\cdot)), \end{cases}$$

which implies that

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)).$$

Therefore, $(u_1(\cdot), u_2(\cdot))$ is also a saddle point of *Problem (ZSG)*.

We define a new *Hamiltonian function* $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{U}_1 \times \mathbb{U}_2 \times \mathbb{R}^n \times \mathbb{R}^{n \times l} \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ as follows:

$$\begin{aligned} H(t, y, z, Y, Z, v_1, v_2, p, \bar{p}, q, \bar{q}) &\triangleq \langle q, f(y, z, Y, Z, v_1, v_2) \rangle + \langle \bar{q}, \bar{f}(y, z, Y, Z, v_1, v_2) \rangle \\ &- \langle p, g(Y, Z, v_1, v_2) \rangle - \langle \bar{p}, \bar{g}(Y, Z, v_1, v_2) \rangle + l(y, z, Y, Z, v_1, v_2). \end{aligned} \quad (28)$$

Let $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the solution $(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot))$ of equation (5). We shall use the abbreviated notation $H(t)$ defined by

$$H(t) \equiv H(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)).$$

The adjoint equations are described by the following generalized stochastic Hamiltonian systems:

$$\begin{cases} dp(t) = -H_Y^*(t)dt - H_Z^*(t)dW(t) - \bar{p}(t)\hat{d}B(t), \\ -dq(t) = H_y^*(t)dt + H_z^*(t)\hat{d}B(t) - \bar{q}(t)dW(t), \\ p(0) = -\varphi_Y^*(Y(0)) - \phi_Y^*(Y(0))q(0), \\ q(T) = \gamma_y^*(y(T)). \end{cases} \quad (29)$$

Based on the above arguments, we can directly derive the following maximum principle for zero-sum games.

Theorem 3.3 (Maximum principle for zero-sum games) *Let (H1) hold and $(u_1(\cdot), u_2(\cdot))$ be a saddle point of Problem (ZSG) with the solutions $(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot))$ and $(p(\cdot), \bar{p}(\cdot), q(\cdot), \bar{q}(\cdot))$ to (5) and (29), respectively. Then it follows that*

$$\left\langle E[H_{v_1}^*(t)|\mathcal{E}_t], v_1(t) - u_1(t) \right\rangle \geq 0 \quad (30)$$

and

$$\left\langle E[H_{v_2}^*(t)|\mathcal{E}_t], v_2(t) - u_2(t) \right\rangle \leq 0 \quad (31)$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, a.e. a.s.

Remark 3.1 If $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point (resp. a saddle point) of nonzero-sum (resp. zero-sum) differential games and $(u_1(t), u_2(t))$ is an interior point of $U_1 \times U_2$ a.s. for all $t \in [0, T]$, then the inequalities in Theorem 3.1 (resp. Theorem 3.3) are equivalent to the following equations

$$E[H_{iv_i}^*(t)|\mathcal{E}_t] = 0, i = 1, 2 \text{ (resp. } E[H_{vj}^*(t)|\mathcal{E}_t] = 0, j = 1, 2).$$

We note that Theorem 3.3 gives a globally necessary condition for a saddle point of zero-sum games. In the following, we begin to present a corresponding locally necessary condition for *Problem (ZSG)*.

We firstly give the following assumptions.

(H3) For all t, τ such that $0 \leq t < t + \tau \leq T$, all bounded \mathcal{E}_t -measurable α_1, α_2 , and for $s \in [0, T]$, the control $\beta_1(s) \doteq (0, \dots, \beta_{1j}(s), \dots, 0)$ and $\beta_2(s) \doteq (0, \dots, \beta_{2j}(s), \dots, 0)$, $j = 1, \dots, n$, with

$$\beta_{1j}(s) \doteq \alpha_{1j} \chi_{[t, t+\tau]}(s) \quad \text{and} \quad \beta_{2j}(s) \doteq \alpha_{2j} \chi_{[t, t+\tau]}(s)$$

belong to \mathcal{U}_1 and \mathcal{U}_2 . For given $u_1, \beta_1 \in \mathcal{U}_1$ and $u_2, \beta_2 \in \mathcal{U}_2$, there exists $\delta > 0$ such that

$$u_1 + \epsilon \beta_1 \in \mathcal{U}_1 \quad \text{and} \quad u_2 + \rho \beta_2 \in \mathcal{U}_2,$$

where β_1 and β_2 are bounded, and $\epsilon, \rho \in (-\delta, \delta)$.

Theorem 3.4 (Local maximum principle for zero-sum games) *Let (H1) and (H3) hold. Let $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the solutions $(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot))$ and $(p(\cdot), \bar{p}(\cdot), q(\cdot), \bar{q}(\cdot))$ to equations (5) and (29), respectively. Further, $(u_1(\cdot), u_2(\cdot))$ is a directional critical point for $J(v_1(\cdot), v_2(\cdot))$, in the sense that, for all bounded $\beta_1 \in \mathcal{U}_1$ and $\beta_2 \in \mathcal{U}_2$, there exists $\delta > 0$ such that $u_1 + \epsilon \beta_1 \in \mathcal{U}_1$ and $u_2 + \rho \beta_2 \in \mathcal{U}_2$, and*

$$h(\epsilon, \rho) \doteq J(u_1 + \epsilon \beta_1, u_2 + \rho \beta_2)$$

has a critical point at $(0, 0)$, for all $\epsilon, \rho \in (-\delta, \delta)$, i.e.

$$\frac{\partial h}{\partial \epsilon}(0, 0) = \frac{\partial h}{\partial \rho}(0, 0) = 0.$$

Then, for all $t \in [0, T]$, we have

$$E[H_{v_1}(t)|\mathcal{E}_t] = E[H_{v_2}(t)|\mathcal{E}_t] = 0. \quad (32)$$

Proof: Since the remaining case can be dealt with by the similar proceeding, we only prove

$$E[H_{v_1}(t)|\mathcal{E}_t] = 0.$$

For $\frac{\partial h}{\partial \epsilon}(0, 0) = 0$, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial \epsilon} h(\epsilon, 0)|_{\epsilon=0} \\ &= E \int_0^T \left(l_y(t) \hat{y}^1(t) + l_z(t) \hat{z}^1(t) + l_Y(t) \hat{Y}^1(t) + l_Z(t) \hat{Z}^1(t) + l_{v_1}(t) \beta_1 \right) dt \\ &\quad + E \left(\varphi_Y(Y(0)) \hat{Y}^1(0) + \gamma_y(y(T)) \hat{y}^1(T) \right). \end{aligned} \quad (33)$$

As shown in Theorem 3.1, applying Itô's formula to $\langle p(t), \hat{Y}^1(t) \rangle$ and $\langle q(t), \hat{y}^1(t) \rangle$, integrating from 0 to T , and substituting them into (33), we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial \epsilon} h(\epsilon, 0)|_{\epsilon=0} \\ &= E \int_0^T \left(q^*(t) f_{v_1}(t) + \bar{q}^*(t) \bar{f}_{v_1}(t) + p^*(t) g_{v_1}(t) + \bar{p}^*(t) \bar{g}_{v_1}(t) + l_{v_1}(t) \right) \beta_1(t) dt \\ &= E \int_0^T H_{v_1}^*(t) \beta_1(t) dt. \end{aligned} \quad (34)$$

In terms of assumption (H3) and equality (34), we further derive that

$$E \int_t^{t+\tau} H_{v_{1j}}(s) \beta_{1j}(s) ds = 0.$$

Differentiating with respect to τ at $\tau = 0$, it yields that

$$E[H_{v_{1j}}(t) \beta_{1j}(t)] = 0. \quad (35)$$

Since equality (35) holds for all bounded \mathcal{E}_t -measurable β_{1j} , we conclude that

$$E[H_{v_1}(t)|\mathcal{E}_t] = 0.$$

Repeating the similar proceeding by differentiating the function $h(0, \rho)$ with respect to ρ , we have

$$E[H_{v_2}(t)|\mathcal{E}_t] = 0.$$

The proof is completed. \square

In the sequel, we give a verification theorem for a saddle point of zero-sum games.

Theorem 3.5 (Verification theorem for zero-sum games) *Let (H_1) hold and $\phi(Y) = MY$ where M is a non-zero constant matrix with order $n \times m$. Let $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the solutions (y, z, Y, Z) and (p, \bar{p}, q, \bar{q}) to equations (5) and (29), respectively. Suppose that the Hamiltonian function H satisfies the following conditional mini-maximum principle:*

$$\begin{aligned} & E \left[H(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)) \mid \mathcal{E}_t \right] \\ &= \inf_{v_1(\cdot) \in \mathcal{U}_1} E \left[H(t, y(t), z(t), Y(t), Z(t), v_1(t), u_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)) \mid \mathcal{E}_t \right] \\ &= \sup_{v_2(\cdot) \in \mathcal{U}_2} E \left[H(t, y(t), z(t), Y(t), Z(t), u_1(t), v_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)) \mid \mathcal{E}_t \right]. \end{aligned} \quad (36)$$

(i) Assume that both φ and γ are concave, and

$$\hat{H}_2(t, a, b, c) = \sup_{v_2(\cdot) \in \mathcal{U}_2} H(t, a, b, c, u_1(t), v_2(t), p(t), q(t), k(t)),$$

exists for all $(t, a, b, c) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, and is concave in (a, b, c) . Then we have

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)), \quad \text{for all } v_2(\cdot) \in \mathcal{U}_2,$$

and

$$J(u_1(\cdot), u_2(\cdot)) = \sup_{v_2(\cdot) \in \mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)).$$

(ii) Assume that both φ and γ are convex, and

$$\hat{H}_1(t, a, b, c) = \inf_{v_1(\cdot) \in \mathcal{U}_1} H(t, a, b, c, v_1(t), u_2(t), p(t), q(t), k(t)),$$

exists for all $(t, a, b, c) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, and is convex in (a, b, c) . Then we have

$$J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)), \quad \text{for all } v_1(\cdot) \in \mathcal{U}_1,$$

and

$$J(u_1(\cdot), u_2(\cdot)) = \inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), u_2(\cdot)).$$

(iii) If both (i) and (ii) are true, then $(u_1(\cdot), u_2(\cdot))$ is a saddle point which implies

$$\begin{aligned} \sup_{v_2(\cdot) \in \mathcal{U}_2} \left(\inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), v_2(\cdot)) \right) &= J(u_1(\cdot), u_2(\cdot)) \\ &= \inf_{v_1(\cdot) \in \mathcal{U}_1} \left(\sup_{v_2(\cdot) \in \mathcal{U}_2} J(v_1(\cdot), v_2(\cdot)) \right). \end{aligned}$$

Proof : (i) Using the same argument as in the proof of Theorem 3.2, we can the following:

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)), \quad \text{for all } v_2(\cdot) \in \mathcal{U}_2. \quad (37)$$

Furthermore

$$\sup_{v_2(\cdot) \in \mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)).$$

Since $u_2(\cdot) \in \mathcal{U}_2$, we have

$$\sup_{v_2(\cdot) \in \mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) = J(u_1(\cdot), u_2(\cdot)).$$

(ii) This statement can be proved in a similar way as shown before.

(iii) If both (i) and (ii) are true, then

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)),$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, i.e. $(u_1(\cdot), u_2(\cdot))$ is a saddle point.

In the following, on the one hand, we have

$$J(u_1(\cdot), u_2(\cdot)) \leq \inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), u_2(\cdot)) \leq \inf_{v_1(\cdot) \in \mathcal{U}_1} \left(\sup_{v_2(\cdot) \in \mathcal{U}_2} J(v_1(\cdot), v_2(\cdot)) \right),$$

and

$$J(u_1(\cdot), u_2(\cdot)) \geq \sup_{v_2(\cdot) \in \mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) \geq \sup_{v_2(\cdot) \in \mathcal{U}_2} \left(\inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), v_2(\cdot)) \right),$$

which imply that

$$\begin{aligned} \sup_{v_2(\cdot) \in \mathcal{U}_2} \left(\inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), v_2(\cdot)) \right) &\leq J(u_1(\cdot), u_2(\cdot)) \\ &\leq \inf_{v_1(\cdot) \in \mathcal{U}_1} \left(\sup_{v_2(\cdot) \in \mathcal{U}_2} J(v_1(\cdot), v_2(\cdot)) \right). \end{aligned} \quad (38)$$

On the other hand, we have

$$J(u_1(\cdot), u_2(\cdot)) \leq \inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), u_2(\cdot)) \leq \sup_{v_2(\cdot) \in \mathcal{U}_2} \left(\inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), v_2(\cdot)) \right)$$

and

$$J(u_1(\cdot), u_2(\cdot)) \geq \sup_{v_2(\cdot) \in \mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) \geq \inf_{v_1(\cdot) \in \mathcal{U}_1} \left(\sup_{v_2(\cdot) \in \mathcal{U}_2} J(v_1(\cdot), v_2(\cdot)) \right),$$

which imply that

$$\begin{aligned} \sup_{v_2(\cdot) \in \mathcal{U}_2} \left(\inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), v_2(\cdot)) \right) &\geq J(u_1(\cdot), u_2(\cdot)) \\ &\geq \inf_{v_1(\cdot) \in \mathcal{U}_1} \left(\sup_{v_2(\cdot) \in \mathcal{U}_2} J(v_1(\cdot), v_2(\cdot)) \right). \end{aligned} \quad (39)$$

Combining (38) and (39), we have

$$\begin{aligned} \sup_{v_2(\cdot) \in \mathcal{U}_2} \left(\inf_{v_1(\cdot) \in \mathcal{U}_1} J(v_1(\cdot), v_2(\cdot)) \right) &= J(u_1(\cdot), u_2(\cdot)) \\ &= \inf_{v_1(\cdot) \in \mathcal{U}_1} \left(\sup_{v_2(\cdot) \in \mathcal{U}_2} J(v_1(\cdot), v_2(\cdot)) \right). \end{aligned}$$

Remark 3.2 *Similar to the results in Section 3.1, we can also give the corresponding corollaries for maximum principle and verification theorem of a saddle point of full information zero-sum differential games. For simplicity, we omit them here.*

4 An example on a nonzero-sum game

In this section, an example of nonzero-sum differential games of FBSDEs is worked out to illustrate our theoretical result. Firstly, by applying the maximum principle (see Theorem 3.1), we find a candidate equilibrium point. Then we obtain the explicit expression of the candidate equilibrium point by virtue of certain filtering techniques of forward-backward stochastic differential equations. Finally, using the verification theorem of an equilibrium point (see Theorem 3.2), we confirm that it is indeed an equilibrium point.

Example: Consider the system of FBDSDE

$$\left\{ \begin{aligned} -dY^{v_1, v_2}(t) &= [a_0(t) + a_1(t)Y^{v_1, v_2}(t) + a_2(t)Z^{v_1, v_2}(t) + F_{i2}(t)v_1(t) + a_4(t)v_2(t)]dt \\ &\quad + b_0(t)\hat{d}B(t) - Z^{v_1, v_2}(t)dW(t), \\ dy^{v_1, v_2}(t) &= [c_0(t) + c_1(t)y^{v_1, v_2}(t) + c_2(t)Y^{v_1, v_2}(t) + c_3(t)Z^{v_1, v_2}(t)]dt \\ &\quad + d_0(t)dW(t) - z^{v_1, v_2}(t)\hat{d}B(t), \\ Y^{v_1, v_2}(T) &= \xi, \quad y^{v_1, v_2}(0) = MY^{v_1, v_2}(0), \end{aligned} \right. \quad (40)$$

with the performance criterion, for $i = 1, 2$,

$$\begin{aligned}
J_i(v_1(\cdot), v_2(\cdot)) = & -\frac{1}{2}\mathbb{E}\left[\int_0^T \left(\langle e_{i1}(t)y^{v_1, v_2}(t), y^{v_1, v_2}(t) \rangle + \langle e_{i2}(t)z^{v_1, v_2}(t), z^{v_1, v_2}(t) \rangle \right. \right. \\
& + \langle e_{i3}(t)Y^{v_1, v_2}(t), Y^{v_1, v_2}(t) \rangle + \langle e_{i4}(t)Z^{v_1, v_2}(t), Z^{v_1, v_2}(t) \rangle + \langle e_{i7}(t)v_i(t), v_i(t) \rangle \Big) dt \\
& \left. \left. + \langle e_{i5}(T)y^{v_1, v_2}(T), y^{v_1, v_2}(T) \rangle + \langle e_{i6}Y^{v_1, v_2}(0), Y^{v_1, v_2}(0) \rangle \right] \right]. \tag{41}
\end{aligned}$$

Here, we assume that all the coefficients in (40) and (41) are bounded and deterministic functions of t , e_{i1}, \dots, e_{i6} are symmetric nonnegative definite, and e_{i7} is symmetric uniformly positive definite. The set of admissible controls is defined by

$$\begin{aligned}
\mathcal{U}_i = \{v_i(\cdot) \mid v_i(\cdot) \text{ is an } \mathbb{R}^{k_i}\text{-valued } \mathcal{E}_t\text{-adapted process} \\
\text{and satisfies } \mathbb{E} \int_0^T v_i^2(t) dt < \infty\}, i = 1, 2, \tag{42}
\end{aligned}$$

Where

$$\mathcal{E}_t = \mathcal{N} \vee \sigma\{W(r) : 0 \leq r \leq t\}.$$

For simplicity, we only deal with the case of 1-dimensional coefficients. Our problem is to find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \sup_{v_1(\cdot) \in \mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \sup_{v_2(\cdot) \in \mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)). \end{cases} \tag{43}$$

Solving: We find the equilibrium point by three steps.

(i) Seek candidate equilibrium points.

Let $\tilde{q}_i(t)$ denote the filtering of $q_i(\cdot)$ with respect to \mathcal{E}_t , i.e. $\tilde{q}_i(t) = \mathbb{E}(q_i(t) | \mathcal{E}_t)$. The similar notations are made for $\tilde{\bar{q}}_i(t), \tilde{p}_i(t), \tilde{\bar{p}}_i(t), \dots, i = 1, 2$.

$$\begin{aligned}
H_i(t, y, z, Y, Z, v_1, v_2, p_i, \bar{p}_i, q_i, \bar{q}_i) \triangleq & \langle q_i, c_0(t) + c_1 y + c_2(t) Y + c_3(t) Z \rangle + \langle \bar{q}_i, d_0(t) \rangle \\
& - \langle p_i, a_0(t) + a_1(t) Y + a_2(t) Z + F_{i2}(t) v_1 + a_4(t) v_2 \rangle - \langle \bar{p}_i, b_0(t) \rangle \\
& - \frac{1}{2} \left(\langle e_{i1}(t) y, y \rangle + \langle e_{i2}(t) z, z \rangle + \langle e_{i3}(t) Y, Y \rangle + \langle e_{i4}(t) Z, Z \rangle + \langle e_{i7}(t) v_i, v_i \rangle \right). \tag{44}
\end{aligned}$$

Applying the maximum principle for nonzero-sum games (Theorem 3.1), we confirm that the

candidate equilibrium points must satisfy the following form:

$$\begin{cases} u_1(t) = -e_{17}^{-1}(t)F_{i2}(t)\tilde{p}_1(t), \\ u_2(t) = -e_{27}^{-1}(t)a_4(t)\tilde{p}_2(t), \end{cases} \quad (45)$$

where $(p_i(\cdot), \bar{p}_i(\cdot), q_i(\cdot), \bar{q}_i(\cdot))$, for $i = 1, 2$, is the solution of the adjoint FBDSDE:

$$\begin{cases} dp_i(t) = \left(e_{i3}(t)Y(t) + a_1p_i(t) - c_2(t)q_i(t) \right) dt \\ \quad + \left(e_{i4}(t)Z(t) + a_2(t)p_i(t) - c_3(t)q_i(t) \right) dW(t) - \bar{p}_i(t)\hat{d}B(t), \\ -dq_i(t) = \left(-e_{i1}(t)y(t) + c_1(t)q_i(t) \right) dt - \left(e_{i2}(t)z(t) \right) \hat{d}B(t) - \bar{q}_i(t)dW(t), \\ p_i(0) = e_{i6}Y(0) - Mq_i(0), \quad q_i(T) = -e_{i5}(T)y(T), \end{cases} \quad (46)$$

and $(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot))$ is the solution of the following equation:

$$\begin{cases} -dY(t) = \left[a_0(t) + a_1(t)Y(t) + a_2(t)Z(t) - F_{i2}(t)^2 e_{17}^{-1}(t)p_1(t) \right. \\ \quad \left. - a_4(t)^2 e_{27}^{-1}(t)p_2(t) \right] dt + b_0(t)\hat{d}B(t) - Z(t)dW(t), \\ dy(t) = \left[c_0(t) + c_1(t)y(t) + c_2(t)Y(t) + c_3(t)Z(t) \right] dt + d_0(t)dW(t) - z(t)\hat{d}B(t), \\ Y(T) = \xi, \quad y(0) = MY(0). \end{cases} \quad (47)$$

(ii) Optimal filtering with the sub-information $\mathcal{E}_t = \mathcal{N} \vee \sigma\{W(r); 0 \leq r \leq t\}$.

Equation (46) together with (47) constitutes a triple dimensional FBDSDE. In order to find the explicit expression of the candidate equilibrium point, we need to compute the optimal filters $\tilde{p}_1(\cdot)$ and $\tilde{p}_2(\cdot)$ of $p_1(\cdot)$ and $p_2(\cdot)$. Applying the filtering result derived by Xiong ([31], Lemma 5.4) to (46) and (47) under the available sub-information $\mathcal{E}_t = \mathcal{N} \vee \sigma\{W(r); 0 \leq r \leq t\}$, we

conclude that $\tilde{p}_1(\cdot)$ and $\tilde{p}_2(\cdot)$ satisfy the following triple dimensional FBSDE:

$$\left\{ \begin{aligned} -d \begin{pmatrix} \tilde{Y}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} &= \left\{ \begin{pmatrix} a_1(t) & 0 & 0 \\ 0 & c_1(t) & 0 \\ 0 & 0 & c_1(t) \end{pmatrix} \begin{pmatrix} \tilde{Y}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right. \\ &\quad + \begin{pmatrix} 0 & -F_{i2}(t)^2 e_{17}^{-1}(t) & -a_4(t)^2 e_{28}^{-1}(t) \\ -e_{11}(t) & 0 & 0 \\ -e_{21}(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{y}(t) \\ \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} \\ &\quad \left. + \begin{pmatrix} a_2(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{Z}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} + \begin{pmatrix} a_0(t) \\ 0 \\ 0 \end{pmatrix} \right\} dt - \begin{pmatrix} \tilde{Z}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} dW(t) \\ \\ d \begin{pmatrix} \tilde{y}(t) \\ \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} &= \left\{ \begin{pmatrix} c_2(t) & 0 & 0 \\ e_{13}(t) & -c_2(t) & 0 \\ e_{23}(t) & 0 & -c_2(t) \end{pmatrix} \begin{pmatrix} \tilde{Y}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right. \\ &\quad + \begin{pmatrix} c_1(t) & 0 & 0 \\ 0 & a_1(t) & 0 \\ 0 & 0 & a_1(t) \end{pmatrix} \begin{pmatrix} \tilde{y}(t) \\ \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} \\ &\quad \left. + \begin{pmatrix} c_3(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{Z}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} + \begin{pmatrix} c_0(t) \\ 0 \\ 0 \end{pmatrix} \right\} dt \\ &\quad + \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -c_3(t) & 0 \\ 0 & 0 & -c_3(t) \end{pmatrix} \begin{pmatrix} \tilde{Y}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ e_{14}(t) & 0 & 0 \\ e_{24}(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{Z}(t) \\ \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_2(t) & 0 \\ 0 & 0 & a_2(t) \end{pmatrix} \begin{pmatrix} \tilde{y}(t) \\ \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} d_0(t) \\ 0 \\ 0 \end{pmatrix} \right\} dW(t), \\ \\ \begin{pmatrix} \tilde{Y}(T) \\ \tilde{q}_1(T) \\ \tilde{q}_2(T) \end{pmatrix} &= \begin{pmatrix} \mathbb{E}[\xi|\mathcal{E}_T] \\ -e_{15}(T)\tilde{y}(T) \\ -e_{25}(T)\tilde{y}(T) \end{pmatrix}, \quad \begin{pmatrix} \tilde{y}(0) \\ \tilde{p}_1(0) \\ \tilde{p}_2(0) \end{pmatrix} = \begin{pmatrix} M & 0 & 0 \\ e_{16}(t) & -M & 0 \\ e_{26}(t) & 0 & -M \end{pmatrix} \begin{pmatrix} \tilde{Y}(0) \\ \tilde{q}_1(0) \\ \tilde{q}_2(0) \end{pmatrix}. \end{aligned} \right. \quad (48)$$

Just like Huang et al.[6], we call (48) a forward-backward stochastic differential filtering equation, which is distinguished from the classical filtering literature (see e.g. Liptser and Shiryaev[8], Xiong[31]). Now, we obtain an explicit candidate equilibrium point for the foregoing LQ nonzero-sum differential game.

(iii) Verify that $(u_1(\cdot), u_2(\cdot))$ denoted by (45) is indeed an equilibrium point.

We can check that the system (40) and performance criterion (41) satisfy the assumptions (H1) and (H2), the Hamiltonian H_i ($i = 1, 2$) denoted by (44) satisfies the conditions (18–20). Then, from Theorem 3.2, we conclude that $(u_1(\cdot), u_2(\cdot))$ denoted by (45) is indeed an equilibrium point.

5 Conclusion

We are concerned with a new type of stochastic differential game problems of FBDSDEs. There are two distinguishing features: One is that game systems are initial coupled; The other one is that differential games is under partial information. We established a maximum principle and a verification theorem, also called a necessary condition and a sufficient condition, for an equilibrium point of partial information nonzero-sum stochastic differential games. Zero-sum games can be considered as a particular case of nonzero-sum games, so we also gave the corresponding conditions for a saddle point of zero-sum stochastic differential games. Finally, we worked out an LQ example and gave the explicit expression of an equilibrium point of nonzero-sum differential games.

It is worth pointing out that game system of FBDSDE covers many cases as its particular case. If we drop its the terms of backward Itô's integral or forward equation or both them , FBDSDE can be reduced to FBSDE or BDSDE or BSDE. Moreover, if we suppose that $\mathcal{E}_t = \mathcal{F}_t$ for all $t \in [0, T]$, all the results are reduced to the case of full information. In addition, stochastic control problems can be regarded as zero-sum stochastic differential games with only one player. Then, our results are a partial extension to Xiao and Wang[30] for optimal control of FBSDEs with partial information, Han et al.[4] for optimal control of BDSDEs with full information, Huang et al.[6] for optimal control of BSDEs with partial information, Wang and Yu[25] and

Yu and Ji[34] for differential games of BSDEs with full information, and Wang and Yu[26] for differential games of BSDEs with partial information. In our game systems of FBDSDEs with partial information, the forward equations are coupled with the backward equations at initial time, not terminal time. So they do not cover each other between our results and those of terminal coupled forward-backward stochastic systems with full or partial information derived by Buckdahn and Li[2], Hamadène[3], Hui and Xiao[7], Meng[11, 12], Øksendal and Sulem[14], Peng and Wu[20], Shi and Wu[21, 22], Wang and Wu[23, 24], Wu[27, 28], Xiao and Wang[29], Zhang and Shi[36], Zhu et al.[37].

Finally, since there are many partial information optimization and game problems in finance and economics, we hope that the results have applications in these areas.

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